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METHOD OF INITIAL FUNCTIONS FOR AXIALLY SYMMETRIC ELASTIC BODIES

S. M. SARGAND and H. H. CHEN
Department of Civil Engineering, Ohio University, Athens, OH 45701-2979, U.S.A.

and

Y. C. DAS
Department of Civil and Mechanical Engineering, McNeese State University, Lake Charles, LA 70609, U.S.A.

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Abstract—Method of initial functions for an axially symmetric state of stress in elasto-dynamic problems has been formulated which gives a complete choice in prescribing the boundary conditions in terms of either stresses or displacements, or a combination of stresses and displacements. The general dynamic response of the elastic body has been derived in the form of a set of transcendental partial differential equations from which the initial functions can be evaluated in terms of prescribed boundary conditions. The method is applied, as an illustration, to the flexural theory of circular plate subjected to antisymmetric lateral loads. Numerical examples of free vibrations of circular plates are given. The results are compared with solutions from classical theory.

INTRODUCTION

The method of initial functions for static problems has been formulated by Vlasov (1957), Vlasov and Leontev (1966) and Lure (1964). Das and Setlur (1970) extended this method to plane elasto-dynamic problems. Later on, this method was also extended to three-dimensional elasto-dynamic problems by Rao and Das (1977). Significant contributions by Iyenger *et al.* (1974a,b, 1975, 1976), Bufler (1971), Bufler and Meier (1975), Haydl (1971a,b), and Haydl and Sherbourne (1976a,b) were made in the areas of solid mechanics and modern control theory prior to the work by Rao and Das.

In the present work, a method has been developed for the dynamics of axially symmetric elastic bodies. The governing dynamic equations for an axially symmetric state of stress and the corresponding constitutive relations are expressed in four first order coupled equations in two displacement and stress components. These equations are taken in the form of a Maclaurin series in the spatial coordinate in the direction of the axis of symmetry, involving functions and their derivatives of stress and displacement on a specified initial plane which is perpendicular to the axis of symmetry. For an elastic body bounded by two such parallel planes two of the four functions will generally be known on each of these bounding planes. Satisfaction of these boundary conditions will result in two transcendental partial differential equations involving the unknown functions on the initial plane or alternatively one equation in an auxiliary function. The stresses and displacements at any layer within the body are expressed in terms of linear combinations of the initial functions and their derivatives.

The method developed here is applied to derive, in a manner that is independent of Kirchhoff's hypothesis, an exact dynamic equation for the axially symmetric oscillations of a circular plate. For straight-crested waves, the velocities of wave propagation are computed for values of wave lengths and the results are compared with those obtained using classical plate theory. The fundamental natural frequency for a circular plate with a clamped edge is also computed using the first approximate theory and is compared with the corresponding value using the classical plate theory.

FORMULATION OF THE PROBLEM

In axially symmetric problems, four initial functions are sufficient to determine the state of stress and strain in the body. These functions are the displacement components $U_0(r,t)$ and $W_0(r,t)$ and the stress components $\sigma_z(r,t)$ and $\tau_{zr}(r,t)$ on the plane $z = 0$ as shown in Fig. 1.

The dynamic equations of an axially symmetric body without body forces are:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\sigma_r - \sigma_\theta}{r} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \tau_{rz}}{\partial r} + \frac{\partial \sigma_z}{\partial z} + \frac{\tau_{rz}}{r} = \rho \frac{\partial^2 w}{\partial t^2}. \tag{1}$$

The constitutive relations are:

$$\sigma_r = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\frac{\partial u}{\partial r} + \nu\frac{\partial w}{\partial z} + \nu\frac{u}{r}\right]$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\frac{\partial u}{\partial r} + (1-\nu)\frac{\partial w}{\partial z} + \nu\frac{u}{r}\right]$$

$$\sigma_\theta = \frac{E}{(1+\nu)(1-2\nu)}\left[\nu\frac{\partial u}{\partial r} + \nu\frac{\partial w}{\partial z} + (1-\nu)\frac{u}{r}\right]$$

$$\tau_{rz} = \frac{E}{(1+\nu)(1-2\nu)}\left(\frac{1-2\nu}{2}\cdot\gamma_{rz}\right) = G\left(\frac{\partial w}{\partial r} + \frac{\partial u}{\partial z}\right). \tag{2}$$

Let

$$u = \frac{1}{G}\frac{\partial U}{\partial r}, \quad \tau_{rz} = \frac{\partial T}{\partial r}, \quad \sigma_z = Z, \quad w = \frac{1}{G}W,$$

$$\alpha^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right), \quad \beta = \frac{\partial}{\partial z}, \quad \xi^2 = \frac{\rho}{G}\frac{\partial^2}{\partial t^2} = \frac{1}{c_s^2}\frac{\partial^2}{\partial t^2}. \tag{3}$$

Introducing the symbols in eqns (3) into eqns (1) and (2), rearranging and using matrix notations, we have:

$$\beta\begin{Bmatrix} U \\ Z \end{Bmatrix} = [A]\begin{Bmatrix} W \\ T \end{Bmatrix}, \quad \beta\begin{Bmatrix} W \\ T \end{Bmatrix} = [B]\begin{Bmatrix} U \\ Z \end{Bmatrix}. \tag{4}$$

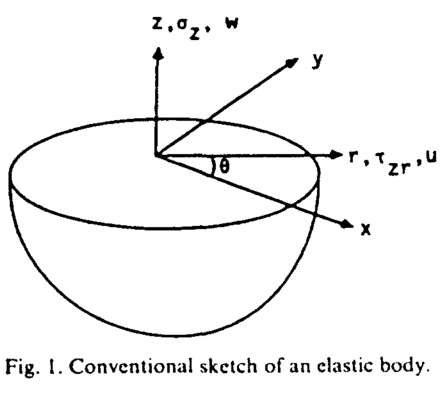

Fig. 1. Conventional sketch of an elastic body.

where

$$[A] = \begin{bmatrix} -1 & 1 \\ \xi^2 & -\alpha^2 \end{bmatrix}, \quad [B] = \begin{bmatrix} \frac{-v}{1-v}\alpha^2 & \frac{1-2v}{2(1-v)} \\ \xi^2 - \frac{2}{1-v}\alpha^2 & \frac{-v}{1-v} \end{bmatrix}. \tag{5}$$

Here, the differential operators, α, β and ξ, follow the usual rules of algebra.

Let us assume the solutions of eqns (4) in a Maclaurin's series in the z direction:

$$\begin{Bmatrix} U(r,z,t) \\ Z(r,z,t) \\ W(r,z,t) \\ T(r,z,t) \end{Bmatrix} = \left(1 + z\beta + \frac{z^2\beta^2}{2!} + \cdots\right) \begin{Bmatrix} U_0 \\ Z_0 \\ W_0 \\ T_0 \end{Bmatrix} \tag{6}$$

where $U_0 = U(r,0,t)$ etc., these being the initial unknown functions on the plane $z = 0$.

The higher derivatives of the initial functions in eqns (5) can be successively obtained by the use of eqns (5). Thus

$$\beta^2 \begin{Bmatrix} U \\ Z \end{Bmatrix} = [C] \begin{Bmatrix} U \\ Z \end{Bmatrix}, \quad \beta^2 \begin{Bmatrix} W \\ T \end{Bmatrix} = [D] \begin{Bmatrix} W \\ T \end{Bmatrix} \tag{7}$$

where

$$[C] = [A][B] = \begin{bmatrix} \frac{v-2}{1-v}\alpha^2 + \xi^2 & \frac{-1}{2(1-v)} \\ \frac{-1}{1-v}\alpha^2\xi^2 + \frac{2}{1-v}\alpha^2 & \frac{1-2v}{2(1-v)}\xi^2 + \frac{v}{1-v}\alpha^2 \end{bmatrix}$$

$$[D] = [B][A] = \begin{bmatrix} \frac{v}{1-v}\alpha^2 + \frac{1-2v}{2(1-v)}\xi^2 & \frac{-1}{2(1-v)}\alpha^2 \\ \frac{-1}{1-v}\xi^2 + \frac{2}{1-v}\alpha^2 & \frac{v-2}{1-v}\alpha^2 + \xi^2 \end{bmatrix}. \tag{8}$$

It can be seen from eqns (7) that even powers of β operating on U, Z or W, T will result in expressions U, Z or W, T respectively. The higher even derivatives in the z direction are obtained by the repeated use of eqns (7) and the higher odd derivatives are obtained by the use of eqns (4) together with eqns (7).

Grouping even and odd powers of β in eqn (6) and substituting from eqns (4) and (7), we get

$$\begin{Bmatrix} U \\ Z \end{Bmatrix} = \left(I + \frac{z^2}{2!}C + \frac{z^4}{!}C^2 + \cdots\right)\begin{Bmatrix} U_0 \\ Z_0 \end{Bmatrix} + A\left(zI + \frac{z^3}{3!}D + \frac{z^5}{5!}D^2 + \cdots\right)\begin{Bmatrix} W_0 \\ T_0 \end{Bmatrix}$$

$$\begin{Bmatrix} W \\ T \end{Bmatrix} = \left(I + \frac{z^2}{2!}D + \frac{z^4}{4!}D^2 + \cdots\right)\begin{Bmatrix} W_0 \\ T_0 \end{Bmatrix} + B\left(zI + \frac{z^3}{3!}C + \frac{z^5}{5!}C^2 + \cdots\right)\begin{Bmatrix} U_0 \\ Z_0 \end{Bmatrix} \tag{9}$$

where I is a 2×2 unit matrix.

Using Sylvester's theorem (Gantmacher, 1980), we can express the function, $F(M)$, of a second order matrix M in the following manner:

$$F(M) = \frac{M-\lambda_2 I}{\lambda_1-\lambda_2}F(\lambda_1)+\frac{M-\lambda_1 I}{\lambda_2-\lambda_1}F(\lambda_2) \tag{10}$$

where λ_1 and λ_2 are the distinct roots of the matrix M.

The roots of the matrices C and D are:

$$\lambda_1 = -\alpha^2+\xi^2, \quad \lambda_2 = -\alpha^2+\Omega\xi^2 \tag{11}$$

where

$$\Omega = \frac{1-2\nu}{2(1-\nu)}.$$

Applying eqn (10) to the functions of matrices involved in eqns (9), we get:

$$C^n = \lambda_1^n C_1+\lambda_2^n C_2, \quad D^n = \lambda_1^n D_1+\lambda_2^n D_2 \tag{12}$$

where

$$C_1 = \frac{1}{\xi^2}\begin{bmatrix} -2\alpha^2+\xi^2 & -1 \\ 2\alpha^2(2\alpha^2-\xi^2) & 2\alpha^2 \end{bmatrix}, \quad C_2 = \frac{1}{\xi^2}\begin{bmatrix} 2\alpha^2 & 1 \\ -2\alpha^2(2\alpha^2-\xi^2) & -2\alpha^2+\xi^2 \end{bmatrix}$$

$$D_1 = \frac{1}{\xi^2}\begin{bmatrix} 2\alpha^2 & -\alpha^2 \\ 2(2\alpha^2-\xi^2) & -2\alpha^2+\xi^2 \end{bmatrix}, \quad D_2 = \frac{1}{\xi^2}\begin{bmatrix} -2\alpha^2+\xi^2 & \alpha^2 \\ -2(2\alpha^2-\xi^2) & 2\alpha^2 \end{bmatrix}. \tag{13}$$

Substituting the proper power of C and D from the general expressions (12) into eqns (9), we get:

$$\begin{Bmatrix} U \\ Z \end{Bmatrix} = (\cos z\gamma_1\cdot C_1+\cos z\gamma_2\cdot C_2)\begin{Bmatrix} U_0 \\ Z_0 \end{Bmatrix}+A\left(\frac{\sin z\gamma_1}{\gamma_1}D_1+\frac{\sin z\gamma_2}{\gamma_2}D_2\right)\begin{Bmatrix} W_0 \\ T_0 \end{Bmatrix}$$

$$\begin{Bmatrix} W \\ T \end{Bmatrix} = (\cos z\gamma_1\cdot D_1+\cos z\gamma_2\cdot D_2)\begin{Bmatrix} W_0 \\ T_0 \end{Bmatrix}+B\left(\frac{\sin z\gamma_1}{\gamma_1}C_1+\frac{\sin z\gamma_2}{\gamma_2}C_2\right)\begin{Bmatrix} U_0 \\ Z_0 \end{Bmatrix} \tag{14}$$

where

$$\gamma_1 = \sqrt{\alpha^2-\xi^2} \quad \text{and} \quad \gamma_2 = \sqrt{\alpha^2-\Omega\xi^2}.$$

The eqns (14) and the expression for σ_n can be written as

$$\begin{Bmatrix} U \\ W \\ Z \\ T \\ \sigma_n \end{Bmatrix} = \begin{bmatrix} L_{uu} & L_{uw} & L_{uz} & L_{ut} \\ L_{wu} & L_{ww} & L_{wz} & L_{wt} \\ L_{zu} & L_{zw} & L_{zz} & L_{zt} \\ L_{tu} & L_{tw} & L_{tz} & L_{tt} \\ L_{nu} & L_{nw} & L_{nz} & L_{nt} \end{bmatrix}\begin{Bmatrix} U_0 \\ W_0 \\ Z_0 \\ T_0 \end{Bmatrix} \tag{15}$$

where

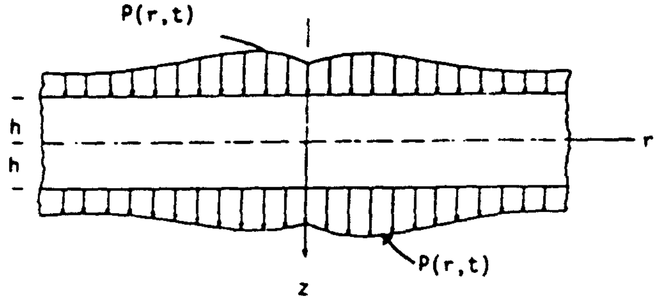

Fig. 2. Circular plate with an axially symmetric load.

$$L_{uu} = L_{tt} = \frac{-2\alpha^2+\xi^2}{\xi^2}\cos z\gamma_1 + \frac{2\alpha^2}{\xi^2}\cos z\gamma_2$$

$$L_{uw} = L_{zt}/\alpha^2 = \frac{2(\alpha^2-\xi^2)}{\gamma_1\xi^2}\sin z\gamma_1 + \frac{-(2\alpha^2-\xi^2)}{\gamma_2\xi^2}\sin z\gamma_2$$

$$L_{uz} = L_{wt}/\alpha^2 = \frac{-1}{\xi^2}\cos z\gamma_1 + \frac{1}{\xi^2}\cos z\gamma_2$$

$$L_{tz} = L_{wu}/\alpha^2 = \frac{(2\alpha^2-\xi^2)}{\gamma_1\xi^2}\sin z\gamma_1 + \frac{-2(\alpha^2-\Omega\xi^2)}{\gamma_2\xi^2}\sin z\gamma_2$$

$$L_{tw} = L_{zu}/\alpha^2 = \frac{2(2\alpha^2-\xi^2)}{\xi^2}\cos z\gamma_1 + \frac{-2(2\alpha^2-\xi^2)}{\xi^2}\cos z\gamma_2$$

$$L_{ww} = L_{zz} = \frac{2\alpha^2}{\xi^2}\cos z\gamma_1 + \frac{-2\alpha^2+\xi^2}{\xi^2}\cos z\gamma_2$$

$$L_{wz} = \frac{a^2}{\gamma_1\xi^2}\sin z\gamma_1 + \frac{-\alpha^2+\Omega\xi^2}{\gamma_2\xi^2}\sin z\gamma_2$$

$$L_{zw} = \frac{-4\alpha^2(\alpha^2-\xi^2)}{\gamma_1\xi^2}\sin z\gamma_1 + \frac{(2\alpha^2-\xi^2)^2}{\gamma_2\xi^2}\sin z\gamma_2$$

$$L_{tu} = \frac{(2\alpha^2-\xi^2)^2}{\gamma_1\xi^2}\sin z\gamma_1 + \frac{-4\alpha^2(\alpha^2-\Omega\xi^2)}{\gamma_2\xi^2}\sin z\gamma_2$$

$$L_{ut} = \frac{-\alpha^2+\xi^2}{\gamma_1\xi^2}\sin z\gamma_1 + \frac{\alpha^2}{\gamma_2\xi^2}\sin z\gamma_2. \quad (16)$$

Equation (15), together with the operators (16), represents in terms of four initial functions U_0, W_0, Z_0, T_0, the complete expressions for the response of an elastic solid in an axially symmetric state. On any plane $z = 0$, any two of these initial functions will be known and the remaining two initial functions have to be solved by using the conditions on any other z = constant plane.

This method is useful in solving a variety of problems in solid mechanics dealing with plates, layered medium, etc. In this paper, the method is illustrated by solving some problems using plate theory.

APPLICATION OF METHOD

Consider an elastic body bounded by two parallel planes and subjected to axially symmetric loads. These loads are antisymmetric on the bounding planes as shown in Fig. 2.

If we take $z = 0$ as the reference plane, $U_0 = Z_0 = 0$ on this plane because of anti-symmetric loading. On the planes $z = +h$, $Z = +P(r,t)$ and $T = 0$. Deleting the terms U_0 and Z_0 in eqn (15) and satisfying the boundary conditions on plane $z = h$ or $z = -h$, we obtain

$$L_{zw}(h)W_0 + L_{zt}(h)T_0 = P(r,t)$$
$$L_{tw}(h)W_0 + L_{tt}(h)T_0 = 0. \quad (17)$$

Introducing an auxiliary function $F(r,t)$ such that

$$L_{tw}(h)\cdot F = -T_0$$
$$L_{tt}(h)\cdot F = W_0. \quad (18)$$

It can be observed that the second of eqns (17) is automatically satisfied and the first of (15) yields

$$[L_{zw}(h)L_{tt}(h) - L_{zt}(h)L_{tw}(h)]\cdot F = P(r,t). \quad (19)$$

By using the operators (16), eqn (19) reduces to

$$\left[\frac{2\alpha^2-\xi^2}{\gamma_2\xi^2}\cos h\gamma_1 \sin h\gamma_2 - \frac{4\alpha^2(\alpha^2-\xi^2)}{\gamma_1\xi^2}\sin h\gamma_1 \cos h\gamma_2\right]F = P(r,t). \quad (20)$$

Operating $L_{tt}(h)$ on eqn (19), we can obtain eqn (20) in terms of the modified transverse deflection of the middle surface W_0.

Equation (20) is the exact transcendental partial differential equation governing the flexural vibrations of a circular plate with axially symmetric loading. This equation has been derived independent of Kirchhoff's hypothesis.

Expanding the trigonometric series in eqn (20) and including terms up to h^3, we obtain the following familiar form:

$$\frac{1}{3}\frac{E}{(1-\nu^2)}h^3\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial w_0}{\partial r}\right)\right] - \frac{2}{3}\frac{(2-\nu)}{(1-\nu)}\rho h^3\left[\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\left(\frac{\partial^2 w_0}{\partial t^2}\right)\right]$$
$$+\frac{7-8\nu}{12(1-\nu)}h^3\frac{\rho^2}{G}\left(\frac{\partial^4 w_0}{\partial t^4}\right) + \rho h\left(\frac{\partial^2 w_0}{\partial t^2}\right)$$
$$= \left\{1 - \frac{h^2}{2}(2+\Omega)\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}\right) - \frac{h^2}{2}\frac{\rho}{G}\frac{\partial^2}{\partial t^2}\right\}P(r,t). \quad (21)$$

It can be seen from eqn (21) that the shear deformation and rotary inertial effects are included.

Particular solutions can be obtained. Let us assume that the solution of eqn (20) (without the load terms in the form of plane radially traveling wave) is

$$w_0(r,t) = \cos 2\pi/\lambda(r-ct) \quad (22)$$

where λ is the wave length and c is the velocity of wave propagation. Substituting eqn (22) into eqn (20) we get:

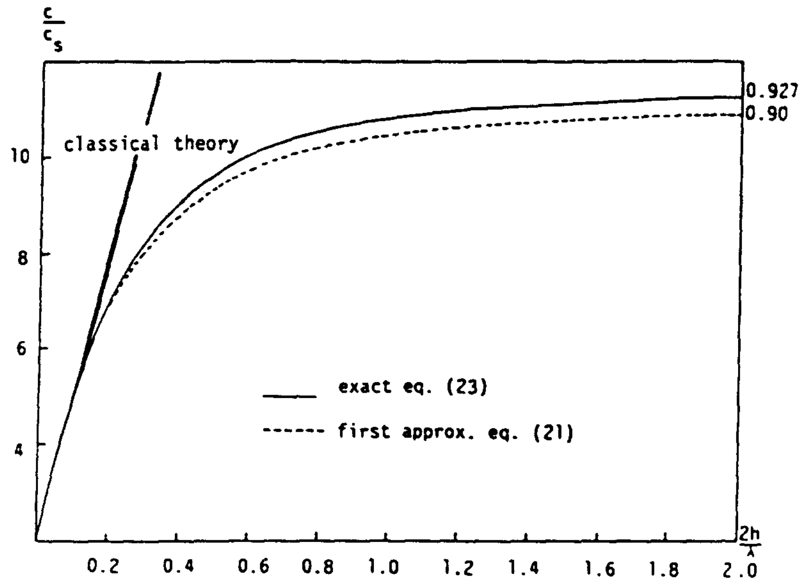

Fig. 3. Variation of c/c_s with $2h/\lambda$.

$$\frac{\tanh\frac{\pi h}{\lambda}\sqrt{1-\Omega\frac{c^2}{c_s^2}}}{\tanh\frac{\pi h}{\lambda}\sqrt{1-\frac{c^2}{c_s^2}}}=\frac{4\sqrt{\left(1-\frac{c^2}{c_s^2}\right)\left(1-\Omega\frac{c^2}{c_s^2}\right)}}{\left(2-\frac{c^2}{c_s^2}\right)^2}. \tag{23}$$

The limiting eqn of (23) as $h/\lambda \to \infty$ is:

$$4\sqrt{\left(1-\frac{c^2}{c_s^2}\right)\left(1-\Omega\frac{c^2}{c_s^2}\right)}=\left(2-\frac{c^2}{c_s^2}\right)^2. \tag{24}$$

Equation (24) is similar to the Rayleigh surface wave equation and gives the lowest value of $c/c_s = 0.927$ for $\nu = 0.3$. The lowest values of c/c_s for various values of h/λ are plotted in Fig. 3 for $\nu = 0.3$ and given in Table 1.

Let us consider axially symmetric flexural oscillations of a circular plate with radius a. By classical theory, the natural frequencies ω are related to the eigenvalues ζ as follows:

Table 1

$2h/\lambda$	c/c_s	
	Exact eqn (23)	First approx. eqn (21)
0.2	0.497	0.486
0.4	0.718	0.691
0.6	0.813	0.779
0.8	0.860	0.822
1.0	0.886	0.847
1.2	0.901	0.861
1.4	0.911	0.871
1.6	0.917	0.877
1.8	0.92	0.882
2.0	0.923	0.885
1000	0.927	0.90

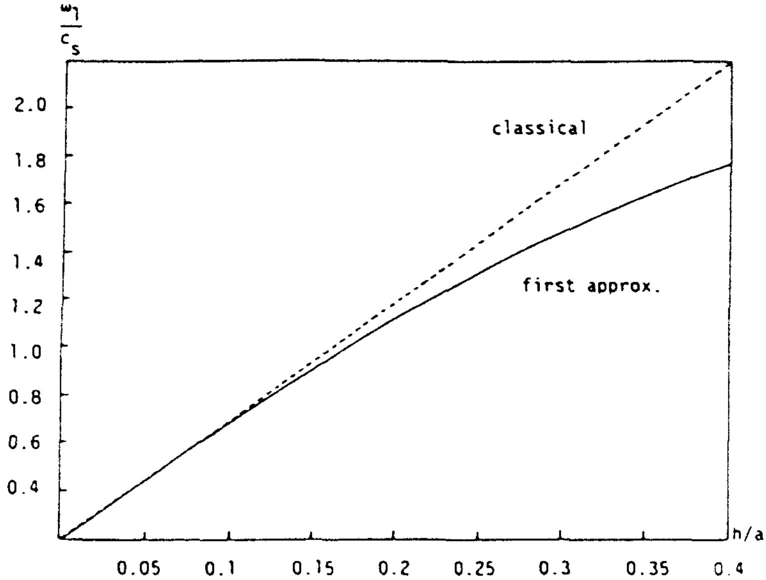

Fig. 4. Variation of ω_1/c_s for various of h/a.

$$\frac{(\pi\zeta)^4}{a^2} = \frac{6(1-\nu)}{(h/a)^2}\left(\frac{\omega}{c_s}\right)^2. \tag{25}$$

Using eqn (21), without external loads, we have:

$$\frac{(\pi\zeta)^4}{a^2} = \frac{6(1-\nu)}{(h/a)^2}\left(\frac{\omega}{c_s}\right)^2 + (2-\nu)(\pi\zeta)^2\left(\frac{\omega}{c_s}\right)^2 - \frac{(7-8\nu)}{8}\left(\frac{\omega}{c_s}\right)^4. \tag{26}$$

For $\nu = 0.3$ and a plate with clamped ends, various values of ω_1/c_s are computed for values of h/a and are plotted in Fig. 4 and given in Table 2.

It can be observed that the fundamental natural frequency obtained by using the classical theory is larger than that obtained by use of the first approximation theory since Kirchhoff's assumption used in deriving the classical theory makes the plate stiffer.

CONCLUSION

The method of initial functions has been developed for an axially symmetric state of stress in an elastic body. Knowing the stresses and displacements on a given plane in the body, the state of stress and displacement can be found at any point in the body by using this method. As an application, rigorous dynamic equations are obtained for the flexural

Table 2

	w/c_s	
h/a	First approx.	Classical
0.05	0.2468	0.2481
0.10	0.4674	0.4961
0.15	0.7125	0.7442
0.20	0.9213	0.9923
0.25	1.1109	1.2404
0.30	1.2809	1.4884
0.35	1.4324	1.7365
0.4	1.5666	1.9846

vibrations of a circular plate and these are in the form of transcendental partial differential equations. Simplified equations of any desired order may be obtained from these equations. Numerical values for fundamental natural frequency of a circular plate with clamped edge are computed by using first approximation theory and compared with similar values computed by using classical theory.

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